

# Emergence of Congestion in Road Networks

Serdar Çolak<sup>1</sup>, Christian M. Schneider<sup>2</sup>, Christian M. Schneider<sup>3</sup> and Marta C. González<sup>4</sup>

<sup>1,2,4</sup> Department of Civil and Environmental Engineering, Massachusetts Institute of Technology, Cambridge, Massachusetts 02319, USA

<sup>3</sup> School of Traffic and Transportation Engineering, Central South University, Hunan 410083, China

<sup>4</sup> Engineering Systems Division, Massachusetts Institute of Technology, Cambridge, Massachusetts 02319, USA

E-mail: [serdarc@mit.edu](mailto:serdarc@mit.edu)

**Abstract.** Particle flows in spatial networks are susceptible to congestion. In this paper, we begin by recalling a framework from previous models of the internet. for analyze the phase transitions of these networks to a state of congested transport and the influence of topology and space on its emergence. The results are confirmed by introducing an analytical solvable framework. We show that the spatial constraints not only affect the critical point, but also change the nature of the transition from a continuous to a discontinuous one. We explore the implications of our findings with an analysis of the San Francisco road network.

## 1. Introduction

Flow networks are inherently liable to congestion. These networks aim to meet demand at reasonable levels as otherwise a congested phase of transport leads to inefficiency and losses. In this sense, the questions of how and where congestion emerges in networks remains significant. Models of the Internet are abundant in physics literature, as flow of data packets are can be captured by simple hopping dynamics [1, 2, 3, 4, 5]. Criticality is a recurring theme as analytical solutions and simulations support its existence [6, 7, 8, 9, 10]. In other analyses, importance of links are evaluated by the use of optimal paths and minimum spanning trees [11, 12, 13]. For this goal, betweenness centrality for shortest path flows is a commonly used metric for identifying critical elements in a network [14, 15].

Models of the Internet rightly overlook the spatiotemporal nature of transport and are not directly applicable to spatial networks, some examples being flows in water reservoir networks [16] and power grids [17]. Transportation and mobility modeling however still remains as the primary area of interest [18]. In this context, capturing traffic flow by making use of the empirically relationships between flow and density in roads is the main methodology [19, 20]. The cell transmission model [21, 22] and the simple point queue models [23, 24, 25] are widely used among such methods. Alternatively, cellular automata models for vehicular traffic [26, 27] along with many other discrete stochastic models [28, 29] have also been analyzed. Our goal in this work is to build on the simple model of the Internet by adding temporal and spatial dimensions of vehicular flow and hence construct a framework to analyze congestion in road networks from a network science perspective.

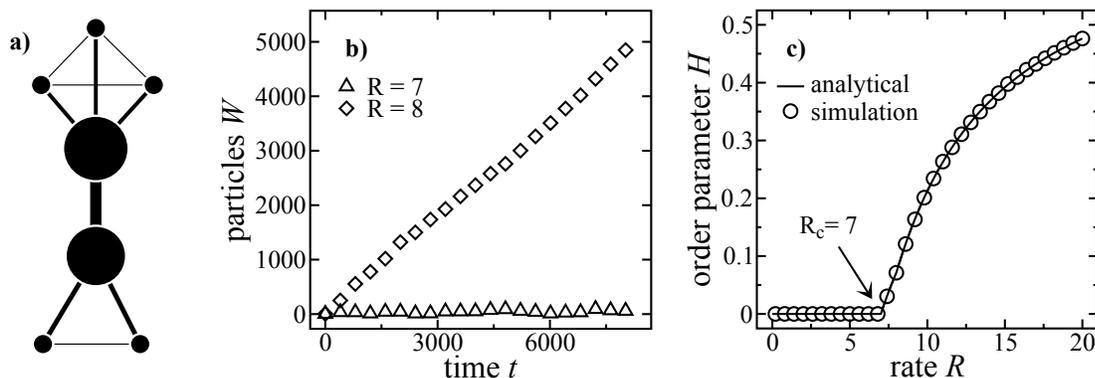
## 2. Models of Transport

### 2.1. Internet Model (IM)

We begin by recalling the model of the internet in [1, 2, 3, 4, 5, 6, 7, 8, 9, 10] (IM): at each timestep  $t$ , the network is loaded with  $R$  identical particles with randomly assigned origins and destinations. A fixed shortest path routing table guides particles towards their destination. Nodes can transmit as many particles per timestep as their *outflow capacity*,  $C$ , and travel between two nodes takes a unit timestep. Queues of particles form at the nodes, and they can grow infinitely large. Particles are exempt from joining the queue at their destination and are removed from the system upon arrival. The network response is measured by the order parameter  $H$  [7]:

$$H(R) = \lim_{t \rightarrow T} \frac{\langle \Delta W \rangle}{R \Delta t}, \quad (1)$$

where  $W$  denotes the number of particles in the system,  $\Delta t$  is the unit timestep and  $T$  is the length of the simulation. Figure 1 depicts the IM and measurements of network response. For relatively low values of  $R$ , the network reaches a rate of particle arrival equal to the loading rate.  $W$  remains constant and consequently  $H = 0$ . If  $R$  exceeds



**Figure 1.** (a) A simple network with node and edge betweenness values mapped as node size and edge width, respectively. (b) Number of particles  $W$  versus time  $t$  for the Internet model for different loading rates  $R$ . (c) A second order transition at the critical value  $R_c = 7$  for this network.

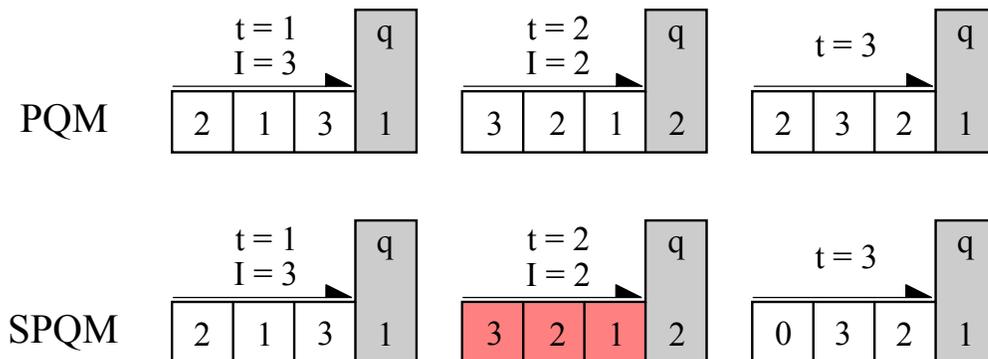
a certain threshold  $R_c$ , a linear increase in  $W$  with a slope of  $H$  is observed due to excessive queueing. The network response exhibits a second-order phase transition.

## 2.2. Queue Models

Spatial networks, for the purposes of this paper, are embedded in two-dimensional space. This property results in non-uniform travel times, differing from models of the Internet where data packets move by hops. Moreover, as these networks carry flows along the links a particle has a specific position on the link it is traveling on. Consequently, particles may also occupy physical space and gradually fill the segment. In this section we present simple queue-based models that capture these properties of spatial flows.

**2.2.1. Point-Queue Model (PQM)** The point-queue model (PQM) [25] is similar to the IM, but differs by shifting flow from nodes to links, incorporating the non-uniform travel time distribution. Links are divided into segments that can be traveled in unit time, and particles travel freely to join a queue at the end of the link from which they will be discharged at the outflow capacity. The total travel time consists of the free travel time and the delay, namely, the timespan between the particle entering the queue and exiting it.

**2.2.2. Spatial Point-Queue Model (SPQM)** The spatial point-queue model (SPQM) has a single additional constraint: an upper limit to the number of particles occupying a link. In this paper, we will refer to this value as the *volume capacity* of the link,  $V$ . Links cannot accept any new particles when they reach their volume capacity, as illustrated in Figure 2. This enables links at volume capacity to clog links upstream and fill them up, causing congestion to spread at rates depending on the loading rate, the network topology and specific link properties among others. Unlike traditional spreading models in the literature the spreading process is gradual. This makes physical analysis



**Figure 2.** Schematic representations of the queue models where numbers in boxes denote the number of particles within that segment of the link. The link has  $C = 2$  for both models and  $V = 8$  for SPQM.  $I$  stands for the inflow expected at the next timestep, and  $q$  represents the queue at the end of the link. Note the different responses at  $t = 3$ : PQM accepts incoming particles whereas SPQM rejects them as volume capacity is reached.

of the model uncommon as movement of non-identical particles along predetermined spatial shortest paths with non-binary spreading are relatively unstudied [30, 31].

The critical loading rate  $R_c^{\text{IM}}$  has been shown [6] to be equal to  $R_c^{\text{IM}} = N(N - 1)(C_{\text{max}}/B_{\text{max}}^N)$  where  $N$  is the network size,  $B_{\text{max}}^N$  is the maximum node betweenness and  $C_{\text{max}}$  is the outflow capacity of this node. This relation arises from the fact that inflow to a node is proportional to its betweenness centrality [32]. At  $R_c$ , the inflow is equal to the outflow at the node with the minimum  $C_{\text{max}}/B_{\text{max}}^N$  value. For the PQM, we adjust this equation by replacing the node betweenness by an edge betweenness value,  $R_c^{\text{PQM}} = N(N - 1)(C_{\text{max}}/B_{\text{max}}^{\text{E}*})$ , where  $B_{\text{max}}^{\text{E}*}$  refers to the maximum *modified betweenness*. This modification is necessary since travelers do not join the queues in the final links, which should be omitted from the edge betweenness calculations.

### 3. Analytical Solutions and Topology

Next, we introduce a framework to analytically calculate the entire transition curve to congestion. For  $R > R_c$ , particle inflow at certain elements will be larger than the outflow. We define  $R_c^i$  as the critical loading rate specific to element  $i$ . For large  $R$ , the outflow of congested links are maximized to capacity, which in consequence affects the inflow to the links downstream. To account for this we define the *delay factor*,  $D_i(R)$ , referring to the fraction of paths through  $i$  that are not suffering from delay as,

$$D_i(R) = \mathcal{H}(R_c^i - R) + \frac{C_i}{I_i(R)} \mathcal{H}(R - R_c^i), \quad (2)$$

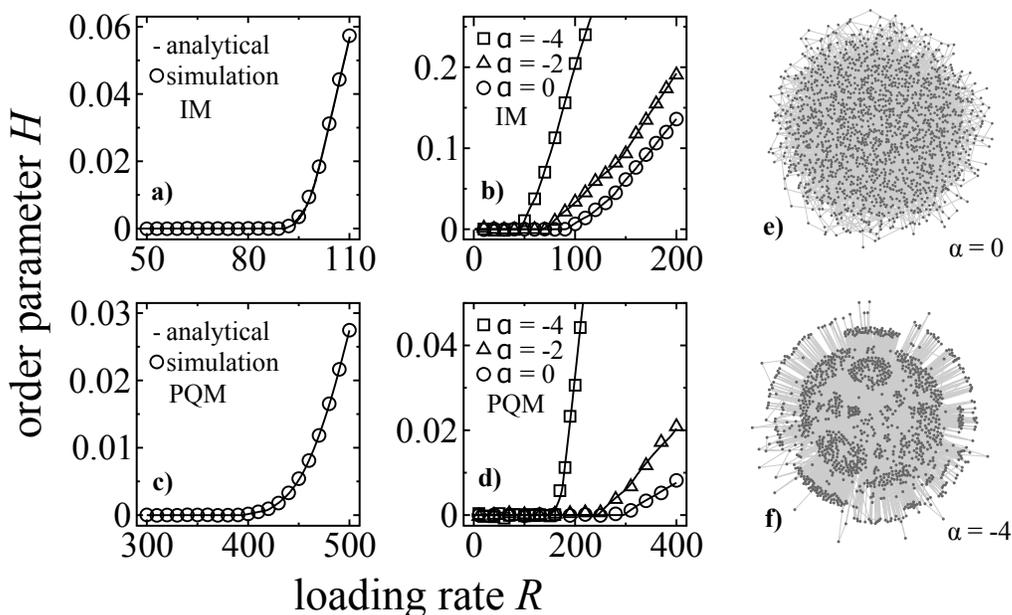
where  $\mathcal{H}(x)$  is the heaviside step function and  $C_i$  and  $I_i(R)$  are the outflow capacity and the inflow of element  $i$  for loading rate  $R$ .  $D_i(R) = 1$  suggests no congestion for element  $i$ , whereas lower values indicate levels of congestion. Using this definition, the

inflow  $I_i(R)$  at a specific loading rate  $R$  can be quantified as,

$$I_i(R) = \frac{\sum_{k \in \Gamma(i)} \prod_{j \in k} R D_j(R)}{N(N-1)} \quad (3)$$

$$H(R) = \sum_{i \in \mathcal{N}} (I_i(R) - C_i) \mathcal{H}(I_i(R) - C_i), \quad (4)$$

where  $\Gamma(i)$  is the set of paths passing through element  $i$ . Eq.(4) accounts for all the delay factors of the elements upstream of element  $i$  by going through the shortest paths. Eq. (2) and (3) form a set of coupled equations that can be solved to obtain the inflows for every element.  $H(R)$  is obtained by summing all positive values of  $I_i(R) - C_i$ .

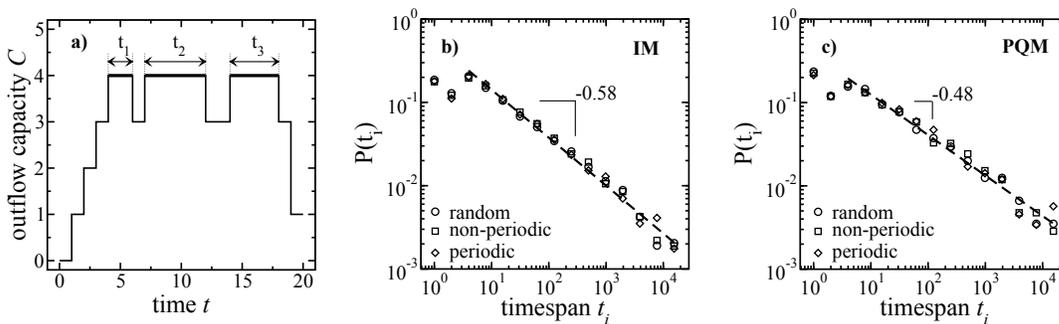


**Figure 3.** (a-d) The transitions for a lattice and its completely rewired instances, showing agreement between simulations and analytical solutions. (e-f) Maps of rewired networks based on connectivity for specified  $\alpha$  values.  $N = 1225$  for all networks.

In order to test different network topologies and examine the effect of space on criticality, we use a non-periodic lattice as a substrate and rewire each edge  $(i, j)$  to a new destination  $j^*$  chosen with probability proportional to  $d(i, j^*)^\alpha$  where  $d(i, j)$  denotes the Euclidean distance between nodes  $i$  and  $j$  [33, 34]. Figure 3 reveals that the simulations and analytical results perfectly coincide for transitions in both the ordered and random networks with size  $N = 1225$  and  $C = 4$ , for both the PQM and the IM. It can be observed that lattices are more resilient to congestion, as transitions show higher critical values. In the rewired instances, lower  $\alpha$  values tend to decrease the critical point and make the transition considerably sharper. Completely rewired versions of the network mapped by connectivity (not in 2-dimensional space) in Figure 3(e) and (f) show that for lower  $\alpha$  values the network becomes more clustered, thus betweenness values increase and critical points decrease.

#### 4. Phase Transition to Congestion

At  $R_c$ , the link that triggers congestion, also referred to as the critical element, is expected to fluctuate between free and congested phases. We investigate the frequency distribution of the timespans at which this most critical element operates at its outflow capacity as an indicator of the temporality of the phase transition. Figure 4 shows that these timespans follow a power law with exponents of  $-0.58 \pm 0.04$  for the IM and  $-0.48 \pm 0.04$  for the PQM, independent of the network topology. This result agrees with the nature of phase transitions where the response undergoes extreme fluctuations at critical points.



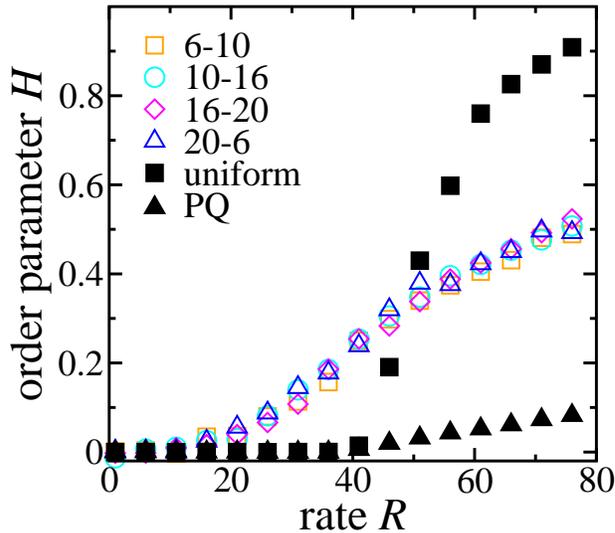
**Figure 4.** (a) Illustration of timespans  $t_i$  at which the critical link operates at its outflow capacity  $C$ . (b) Distributions of the timespans through which the congested element operates at its outflow capacity at  $R_c$  for the (b) IM and the (b) PQM, for the topologies specified.

SPQM results in a different behavior. For low volume capacities links tend to fill up, causing links upstream to fill as well. Fluctuations cause a *gridlock*, a condition where all elements of a cycle are completely filled and hence travel comes to a halt. In case of a gridlock, the order parameter increases very sharply. For large volume capacities, PQM and SPQM have the same  $R_c$ , which we will refer to as the PQM-limit. At steady state, queues are not expected to be necessarily empty but rather steady in their size. If the volume capacity of a link is smaller than this steady state queue size, travelers will be blocked in the upstream link which consequently may suffer from a decrease in its outflow due to this clogging effect. This suggests a lower critical point for a network with active volume capacity constraints, correspondingly  $R_c^{\text{PQM}} \geq R_c^{\text{SPQM}}$ . Therefore, to realize the congestion-free transport to the fullest, the PQM-limit should be aimed.

#### 5. An Analysis of San Francisco

Figure 5 depicts the transitions for the PQM and the SPQM for the San Francisco road network with  $N = 1152$  and  $\langle k \rangle = 3.2$ . The network is discretized by unit travel times of 10 seconds. Outflow capacity of a road segment is obtained by using the speed limit and the number of lanes. Volume capacities are estimated for every road segment assuming that the volume capacity is reached when speed drops to one thirds of the

speed limit. Under these assumptions, the PQM-limit almost reached as  $R_c^{\text{PQM}} = 42$  (11340 vehicles/hr) and  $R_c^{\text{SPQM}} = 40$  (10800 vehicles/hr).



**Figure 5.** The response of the San Francisco road network to theoretical uniform demand and to empirical origin-destination data obtained via mobile phones for the SPQM.

The response is compared to realistic origin-destination matrices obtained by analyzing mobile phone-call records in the Bay Area for 4 intervals of a regular weekday: morning (6am to 10am), noon and afternoon (10am to 4pm), evening (4pm to 8pm) and night [35]. Travel patterns obtained from this data are preserved while tuning the number of trips produced at every timestep,  $R$ . As observed by the overlapping curves, different times of day bring out similar responses from the road network when at the same magnitude within the city scale. Results also show that the critical point shifts to the left, that is, the road network is less resilient to actual distributed demand as opposed to the uniform. However the response is less dramatic as suggested by the slope of the transition. This is caused by the fact that the road network conforms better to actual travel patterns, and is fulfilled more efficiently when loaded to saturation.

## 6. Conclusions

Our findings suggest that the sharper response of the SPQM can be traced back to the inherence of congestion in downtown areas: cities with high population densities have concentrated spatial demand distributions, which is both caused by and result in the inadequacy of urban space to accommodate such concentrated flows. Therefore the interplay between the efficiency of density in urban areas and the excessive load it applies to the road system is further emphasized in this work.

This work is a step further on a systems analysis applied to congestion in roads. In further studies, more detailed population and facility distributions can be modeled within a similar framework. The use of empirical origin-destination matrices that represent the spatial distribution of trips can provide more interesting urban insights on where and how congestion emerges in road networks [35].

## 7. References

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