

OPINION FORMATION ON A DETERMINISTIC PSEUDO-FRACTAL NETWORK

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The Sznajd model of socio-physics, with only a group of people sharing the same opinion can convince their neighbors, is applied to a scale-free random network modeled by a deterministic graph. We also study a model for elections based on the Sznajd model and the exponent obtained for the distribution of votes during the transient agrees with those obtained for real elections in Brazil and India. Our results are compared to those obtained using a Barabási–Albert scale-free network.

Keywords: Social systems; deterministic graphs; small-world networks; scale-free networks; elections.

1. Introduction

Growing interest has been focused on the statistical physics of complex networks. They describe a wide range of systems in nature and society, modeling diverse systems as the Internet, the World Wide Web, the net of human sexual contacts, a network of chemicals linked by chemical reactions, as well as social and biological networks.^{1–6} Such networks exhibit a rich set of scaling properties. A number of them are scale-free, that is, the probability that a randomly-selected node has exactly k links decays as a power law, following $P(k) \sim k^{-\gamma}$, where γ is the degree exponent. Consequently they present resilience against random breakdowns, an effect known as robustness of the network. The shortest-path length between their sites grows slowly (i.e., logarithmically) with the size of the network, that is, compared to the large sizes of networks, the distance between their sites are short — a feature known as the “small world” effect.

The majority of networks used to generate a scale-free topology are stochastic, i.e., the nodes appear to be randomly-connected to each other. These scale-free random networks have naturally a continuous degree distribution. But recently it

has been shown that discrete degree distributions of some deterministic graphs also asymptotically exhibit a power law decay.³ Furthermore, scale-free random networks are excellently modeled by such deterministic graphs.⁴ However the comparison between the behavior of stochastic and deterministic networks in the simulation of a particular model still remains open.

The involvement of physics in research on spreading of opinion has been increasing.^{7,8} Of particular interest here is the Sznajd model,⁹ which is one of several recent consensus-finding models^{8,10} and in which each randomly-selected pair of nearest neighbors convinces all its neighbors of their opinion, only if the pair shares the same opinion (“together we stand”); otherwise, the neighbor opinions are not affected. One time step means that on average every lattice node is selected once as the first member of the pair. It differs from other consensus models by dealing only with communication between neighbors, and the information flows outward, as in rumor spreading. In contrast, in most other models the information flows inward. Initially, two opinions (+1 and -1) are randomly-distributed with probability p and $1 - p$, respectively, over all the nodes of the lattice. The basic Sznajd model with random sequential updating always leads to a consensus (all sites have the same opinion and the whole system reaches a fixed point after a certain simulation time). A phase transition is often observed as a function of the initial concentration of opinion p . A generalization to many different opinions (instead of only ± 1) simulated on a Barabási–Albert network¹¹ reproduced quite well the results of the complex elections of city councillors in the state of Minas Gerais in Brazil in 1998.¹²

In the next section we present the deterministic scale-free network. Then we simulate using the Sznajd model an election process on such a network. We compare our results with the same simulations carried out on a stochastic scale-free network (the Barabási–Albert network) and with states election’s from Brazil and India.

2. Models and Results

The deterministic scale-free graph used in this work grows as follows: At each time step, every edge generates an additional vertex, which is attached to both end vertices of this edge. Initially, at $t = 0$, we have a triangle of edges connecting three vertices, at $t = 1$, the graph consists of six vertices connected by nine edges, and so on (see Fig. 1). The total number of vertices at iteration t is

$$N_t = \frac{3(3^t + 1)}{2}. \quad (1)$$

This simple rule produces a complex growing network. Such a graph is called a *pseudo-fractal*. It has a discrete degree distribution. To relate the exponent of this discrete degree distribution to the standard γ exponent of a continuous degree distribution for random scale-free networks, we use a cumulative distribution $P_{\text{cum}}(k)$, which follows

$$P_{\text{cum}}(k) \sim k^{1-\gamma}, \quad (2)$$

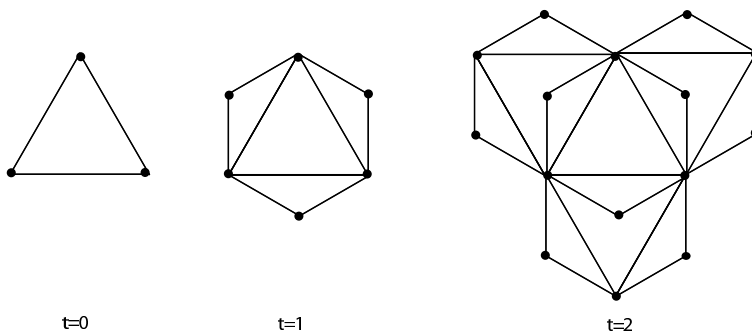


Fig. 1. The three first generations of the scale-free pseudo-fractal graph. At each iteration step t , every edge generates an additional vertex, which is attached to the two vertices of this edge.

and it is the probability that a node of the network has at least k neighbors. It decreases as a power of k with exponent $\gamma = 1 + \ln 3 / \ln 2$. In a similar way the average clustering coefficient,⁴ which is the probability of existence of a link between two nodes when they are both neighbors of a same node, can be calculated for the infinite graph, $\bar{C} = 4/5$. One obtains a shortest-path-length distribution which tends to a Gaussian of width $\sim \sqrt{\ln N}$ centered at $\bar{l} \sim \ln N$ for large networks.⁴ These properties concerning the degree distribution, the clustering coefficient and the mean length, are also present in a wide range of stochastic scale-free networks reported in the literature. They make our simple deterministic networks suitable to examine applications very often found on stochastic networks. In the next section we apply the Sznajd model to our scale-free pseudo-fractal.

2.1. Monte Carlo simulations of the Sznajd model

We let the fractal of Fig. 1 grow and at each step assigns sites with random opinion ± 1 . At every step $t > 0$, we have the following process:

- (1) The network grows, i.e., 3^t new sites are added.
- (2) A random opinion (± 1) is set to each new node of the network, with probability p ($1 - p$) for opinion $+1$ (-1).
- (3) N_s Sznajd runs are performed. For each run, 3^t sites chosen randomly are analyzed and updated, i.e., one visits for the Sznajd model a number of sites equal to the number of sites added at that step to the network.

Three variations of the Sznajd model on the pseudo-fractal network have been investigated:

- **1 site convincing:** For each site i chosen, we change the opinion of all its neighbors to the site's opinion.
- **2 sites convincing:** For each site i chosen, we select randomly one of its neighbors. If this selected neighbor has the same opinion of the site i , then all their neighbors follow the pair's opinion. Otherwise, nothing is done.

- **3 sites convincing:** For each i site chosen, we select two of its neighbors at random. If all these three sites have the same opinion, they change the opinion of all their neighbors.

Figure 2 shows the mean opinion for the three rules (1, 2 and 3 sites convincing) for different initial concentrations p and N_s Sznajd runs per step (rps): 1 rps ($N_s = 1$) corresponds to one realization of Sznajd per step, that is at each step, one chooses 3^t nodes of the network at random to simulate Sznajd. 10 rps ($N_s = 10$) and 50 rps ($N_s = 50$) correspond, respectively, to choose 10×3^t and 50×3^t nodes randomly per step. We let the network grow up to 29 576 sites, which corresponds to 10 steps. For 1 node convincing, the model cannot reach consensus, the results become more unpredictable, the larger the number of runs per step (Fig. 2(a)). However, for two and three nodes convincing, a full consensus is observed (Figs. 2(b) and 2(c)). We clearly see that for rules 2 and 3 one has a 1st-order transition since the order parameter (opinion) jumps drastically and shows strong hysteresis. Our results are very similar to the ones obtained on the Barabási–Albert network when the same rules are applied,¹³ except that the latter requires more Sznajd runs for the network to reach a fixed point, i.e., a full consensus.

Since after some time steps the rules 2 and 3 always lead to a consensus and the whole system reaches a fixed point, in Fig. 3 we show the number of samples, out of 1000, for which the fixed point all “up” (all with opinion +1) is obtained when different values for the initial concentration p of nodes “up” are simulated. As we can see, the fraction of realizations with fixed point “up” depends on the probability p and on the implemented rule. This is opposed to the results obtained with the same rules (2 or more nodes convincing) on a square lattice, where an abrupt change is observed for $p \geq 0.5$;¹⁴ but in agreement with the ones on a Barabási–Albert network¹³ and on a one-dimensional chain.⁹ For 1 node convincing the system does not tend to a fixed point (consensus), while on square lattices it does,^{15,16} although without the abrupt change of 2 sites convincing.

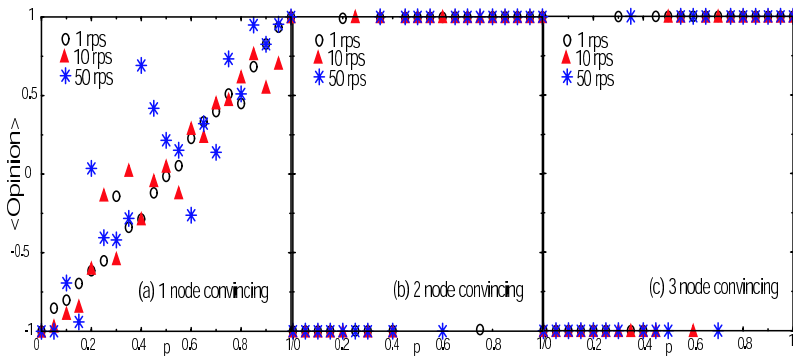


Fig. 2. Sznajd model on a 29 576 nodes pseudo-fractal network with $N_s = 1, 10$ and 50 runs per time step for (a) 1 node convincing, (b) 2 nodes convincing, and (c) 3 nodes convincing.

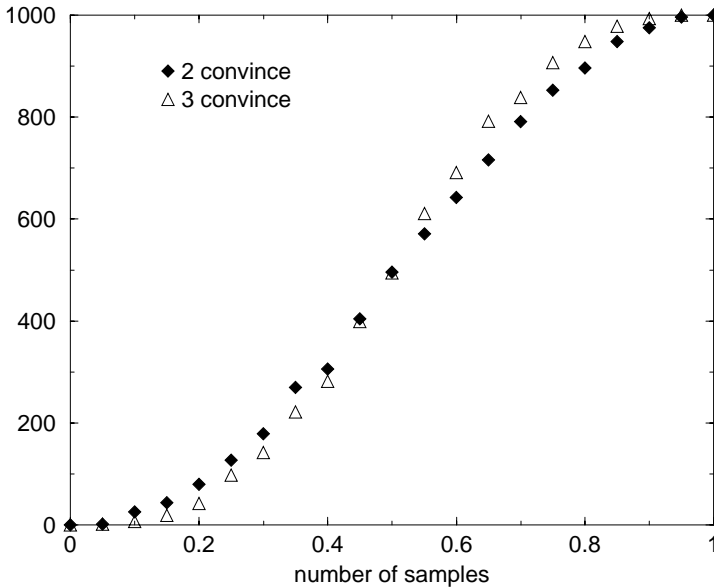


Fig. 3. Monte Carlo simulation on a 29 576 nodes pseudo-fractal network counting the number of samples, out of 1000, for which the fixed point all “up” is obtained when different values for the initial concentration p of nodes “up” are simulated. As we can see, the fraction of realizations with fixed point up depends on the probability p and on the rule implemented, for Sznajd models with 2 and 3 nodes convincing.

2.2. Simulation of elections with many candidates

We create a network of interacting nodes by using the pseudo-fractal network prescription as described before (Eq. (1)). In addition to our network, we also simulated the rule presented in Ref. 12 for modeling elections on a Barabási network. We summarize the generation of the Barabási network as follows: At the beginning of the simulation we have six fully-connected nodes. Then, we add more and more nodes and connect it with five of the present nodes chosen at random. Thus the growth probability at any existing node is proportional to the number of nodes already connected to it. For comparison we generate both networks with the same number of nodes (voters), and the same number of candidates.

After generating a network, one starts with the “election process”. N candidates are randomly-distributed. The value n ($1 \leq n \leq N$) of a node on the network represents that this voter has preference for the candidate n . Initially, all sites start with value zero, meaning that there is no preference for any candidate, except for N sites that have the number of a candidate. Now, the electoral campaign starts (only voters with preference for a candidate can influence other voters, *à la Sznajd*). At each time step all the nodes are randomly-visited: a random list of nodes assures that each node is reached exactly once. For each visit, we implement the following process:

- We choose a node i at random, if it has preference for a candidate, we choose among its connected nodes, a node j at random. Otherwise, we randomly select another node.
- If node j has the same candidate as node i , each node convinces all the nodes connected to it with probability:

$$p(k) = \frac{1}{n(k)^{1/\xi}}, \quad (3)$$

where $n(k)$ is the number of nodes connected to either i or j , and $\xi > 1$ a switching factor that is analyzed later.

- If node j has no candidate, node i convinces it to accept its own candidate with the probability $p(i)$ of Eq. (3).
- If node j has a different candidate from node i , we choose another node i .

As in real elections, we do not wait for a kind of fixed point which here would correspond to all the nodes preferring the same candidate, but we count the votes at an intermediate time. We group in a histogram the *number of candidates* which received a certain number of *votes*. Because the bin size for the *votes* increases by a factor 2 for each consecutive bin we divide each point of the histogram by the bin size. For this reason we have numbers lower than one for the histogram of the number of candidates. This kind of “voting distribution” is used in the literature for analyzing similar results.^{12,17} In Fig. 4 we see that the results of the voting distribution for the simulations on the pseudo-fractal and on the Barabási network agree very well, using $\xi = 2$ (Eq. (3)), for the pseudo-fractal. In the inset we show the results of the simulation on the pseudo-fractal for $\xi = 1, 2, 4, 8$ and 16 , after 400, 20, 4, 2 and 1 iterations, respectively. The results differ for each selection of ξ : the system reaches a fixed point more rapidly for larger parameter ξ , i.e., one needs less iterations in order to reach the same distribution’s width. Using the same value used in the simulations on the Barabási network, $\xi = 1$, after nearly 2000 iterations, a fixed point (consensus) on the pseudo-fractal network could not be reached. A similar result is observed on the square lattice for the Sznajd model, where if convincing happens only with a certain probability, then no complete consensus is found.^{14,20}

For the simulation on the Barabási network we use $\xi = 1$, like reported in Ref. 12; this selection means that on average each node convinces one other node at each process. As we show in Fig. 5 the degree distribution of the pseudo-fractal is discrete and is given by $P_{\text{cum}}(k) = k^{-\ln 3/\ln 2}$, while the degree exponent for the Barabási network presented in this work is $\gamma = 2.9$.¹¹ Knowing $P(k)$ we can calculate the probability distribution of convincing choosing one site at random, that is the probability of convincing Eq. (3) multiplied by the degree distribution $P(k)$. For the convincing distribution of both networks to be similar, one has to chose Eq. (3) with $\xi = 2$ for the simulation on the pseudo-fractal. We obtain qualitatively the same results in both simulations, being 12 time faster in the pseudo-fractal than in

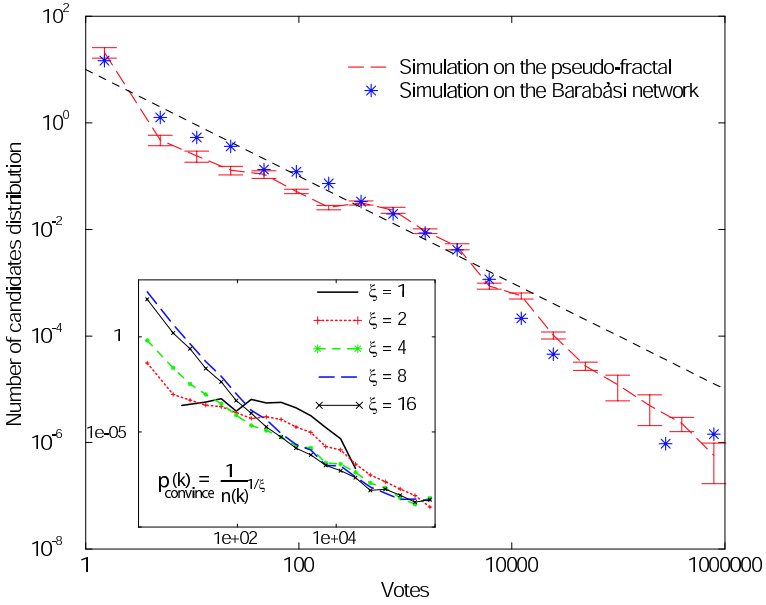


Fig. 4. The plot of the voting distribution of an election process on the pseudo-fractal compared to the simulation on a Barabási network. Both networks have 797 163 sites (voters) and 500 candidates. We average over 20 realizations for each network. The distribution for the pseudo-fractal is after 15 iterations of the convincing process with $\xi = 2$ (Eq. (3)). The distribution for the Barabási network is after 83 iterations. In the inset we show the results of the simulation on the pseudo-fractal for $\xi = 1, 2, 4, 8$ and 16, after 400, 20, 4, 2 and 1 iterations, respectively. We see that the results depend on the rule chosen.

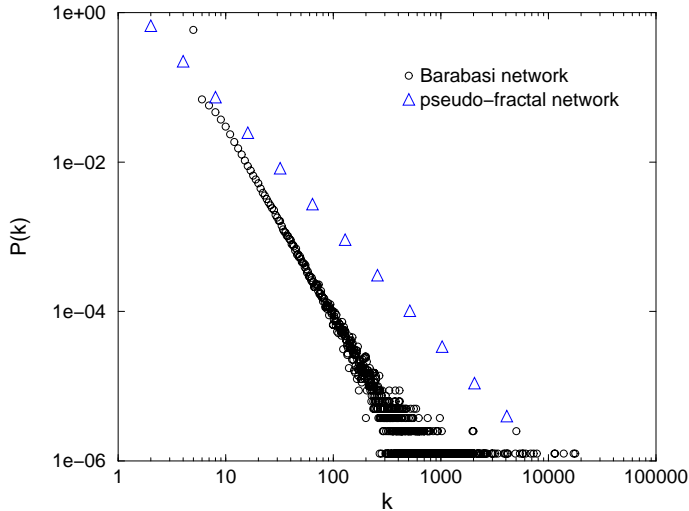


Fig. 5. Comparison of the degree distributions of the Barabási network $P(k) \sim k^{-\gamma}$ ($\gamma = 2.9$), and of the pseudo-fractal network, $P(k) \sim k^{1-\gamma}$ ($\gamma = 1 + \ln 3 / \ln 2$).

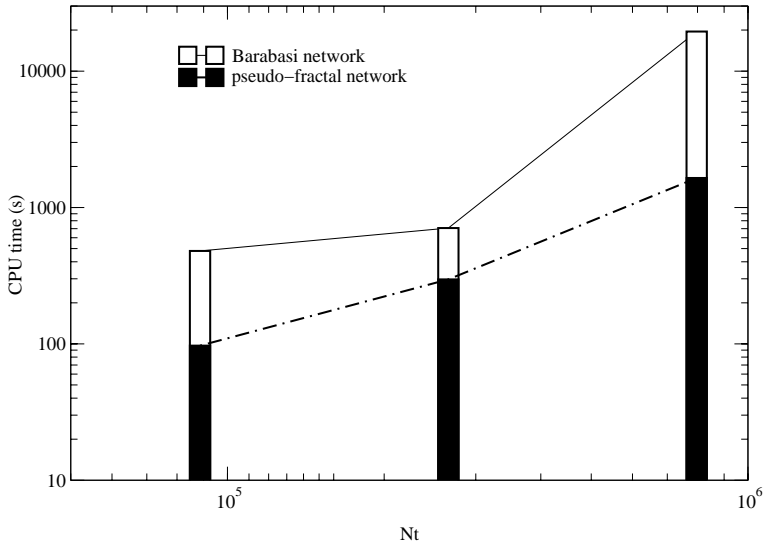


Fig. 6. CPU time (s) on a PIV 2.4 GHz versus size of the network for the pseudo-fractal network and for the Barabási network. We simulate an election process with 100 iterations of the convincing model, on networks of sites: 88 575, 265 722 and 797 163 with 500 candidates. The computation time on the pseudo-fractal grows linearly with the size of the system, while for the stochastic network it grows exponentially.

the Barabási network, because it requires less memory space and computation time (see Fig. 6).

In Fig. 7 we see that the shape of the voting distribution after a given number of iterations does not change with the size of the network. Changing the number of candidates does not alter the form of the distribution (see the inset of Fig. 7).

We consider the results from two Brazilian states (São Paulo and Minas Gerais) for the positions of local state deputies. In such elections the voters vote directly for the candidate and not for parties. Some elections occur with a high number of candidates, of the order of thousands, and with a number of voters of the order of millions. The official results for each state are available on the Internet.¹⁸

In Fig. 8 we see the results of a simulation on a 15-generation pseudo-fractal (21 523 362 nodes) and 1144 candidates, and compare it to the results of real elections described in Fig. 8. For the moment we do not take into account abstentions, or invalid votes in the simulation. The pseudo-fractal tends to consensus in a similar way as the real elections. After averaging over more than 100 realizations we see that the deviations from a perfect line in the intermediate regions of votes, seen in the real elections, are not of statistical nature but seem due to the determinism of the network. The general behavior of the distribution of candidates of the simulation of the elections on the pseudo-fractal however agrees with the one observed in real results. The results of the distribution of candidates for the simulated and

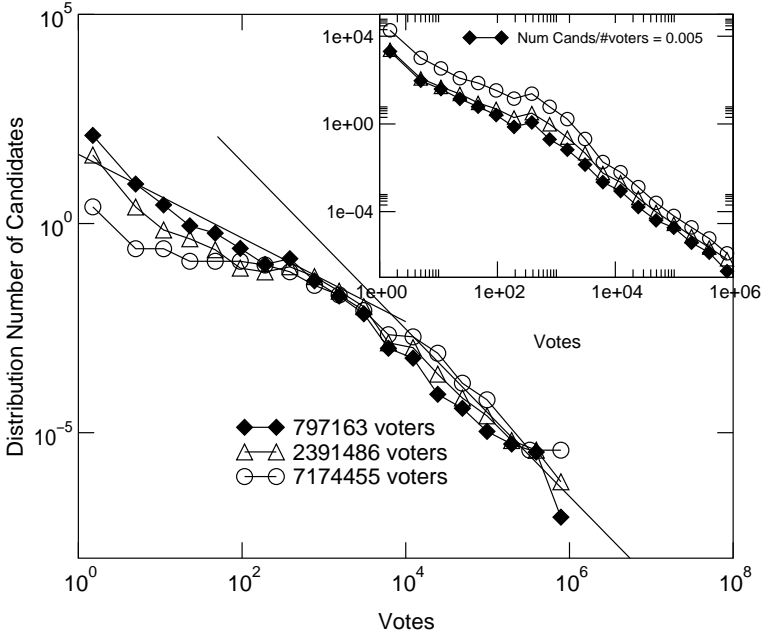


Fig. 7. Test for size effects on the distribution of candidates for the simulation of elections on the pseudo-fractal with 12 generations (\blacklozenge), 13 generations (\triangle) and 14 generations (\circ) with 2000 candidates each, after 15 iterations of the election process. In the inset we make the simulation with a number of candidates proportional to the number of sites of the network, keeping the relation $\#Cands/\#voters = 0.005$, we see that the results are size-independent. The solid lines are guides to the eye with slopes -1 and -2 .

the real cases follow hyperbolic law,

$$N(v) \propto \frac{1}{v}, \quad (4)$$

for the number N of candidates having v votes, extending over two or three order of magnitude, with deviations only for small and large numbers of votes.¹⁷

In addition we analyzed the behavior of the voting distribution for the elections in India to the lower house of the Parliament (Lok Sabha). These elections are events involving political mobility and organizational mobility on an amazing scale. In the 1998 election to Lok Sabha there were 1269 candidates from 38 officially recognized national and state parties seeking election, 1048 candidates from registered parties not recognized and 10 635 independent candidates. A total number of 596 185 335 people voted. The Election Commission employed almost 400 000 people to run the election. The official results of this elections are available on the Internet.¹⁹ In contrast to the Brazilian elections, in this process the country is divided into 543 parliamentary constituencies, each of which return one representative to the Lok Sabha, that is, not all the voters elect among all the candidates of one state (like in Brazil), but there is one election process for each of the 543 representatives, that occurs in each parliamentary constituency. These parliamentary constituencies are

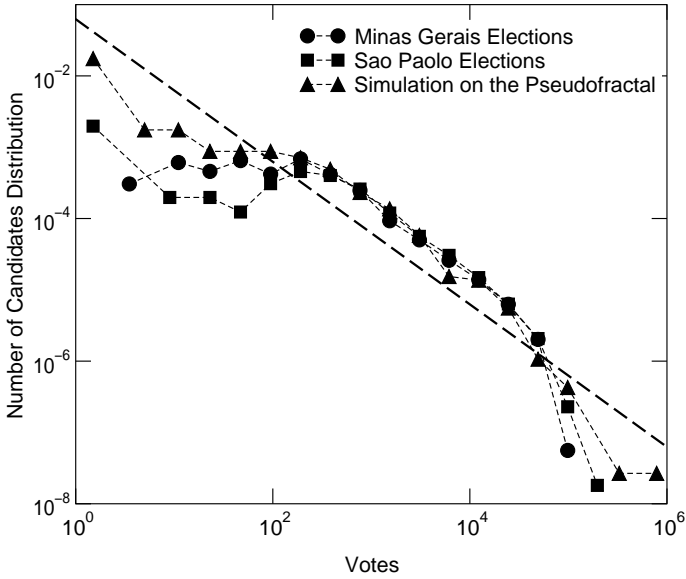


Fig. 8. Result of the simulation of an election process after 20 iterations on a pseudo-fractal network of 21 523 362 nodes and 1144 candidates (\blacktriangle). Compared to the voting distribution for the state of Minas Gerais in 1998 (\bullet) (819 candidates, 11 815 183 voters) and the state of São Paulo (1998) (\blacksquare) (1260 candidates, 23 321 034 voters). Both axis are plotted on a logarithmic scale. The dashed straight line is a guide to the eye with slope -1 . The bin size for the *votes* increases by a factor 2 for each consecutive bin. The height of the distribution of the pseudo-fractal is multiplied by 10 to see better the comparison of the results.

selected by an independent Delimitation Commission, which creates constituencies which have roughly the same population, subject to geographical considerations and the boundary of the states and administrative areas.

At the end of the process one can analyze the voting distribution of each state of the India. In Fig. 9 we observe the results for five states: Uttar Pradesh, Goa, Andhra Pradesh, Haryana and Kerala, with 85, 2, 42, 10 and 20 constituencies, respectively. The voting distribution for each state is the superposition of different electoral processes for all the constituencies of the state. For each election in a constituency there is a voting result that corresponds to few candidates, between 5 and 10. The difference between the India and Brazilian results appears at this stage. If one analyses the election process of each parliamentary constituency, even though one does not have a large number of candidates for the statistics, a distribution with slope ~ -1 is observed. However, the final results of the process for each Indian state provide a different profile far from the hyperbolic one ($1/v$ type of distribution) (see Fig. 9), which was observed for many Brazilian states.^{12,17} In order to analyze the number of candidates N which received a certain fraction of votes v for the nationwide voting process, we have normalized the votes of each candidate by the total number of voters (Fig. 10). As can be seen, the number of candidates N follows a power law $N(v) \propto v^\alpha$, with $\alpha = -1.3$ (for Brazil, $\alpha = -1$).

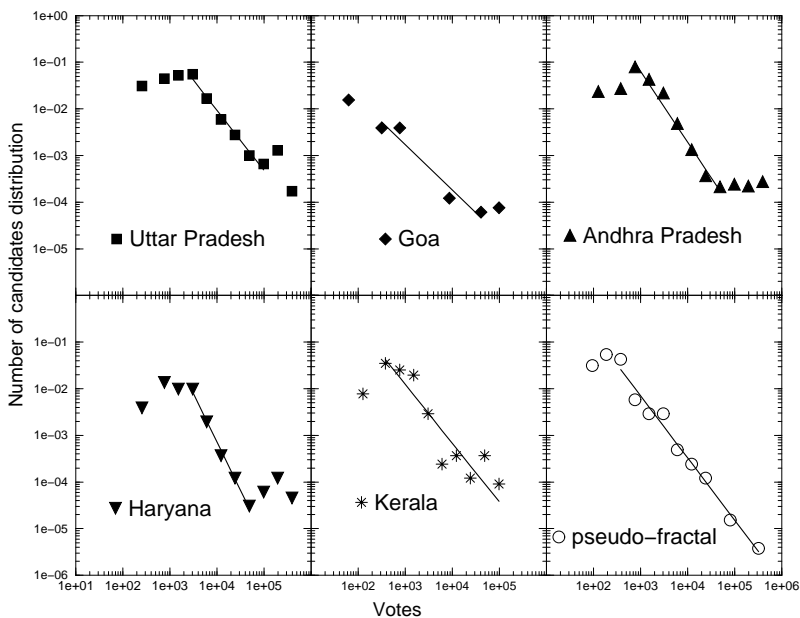


Fig. 9. Voting distribution for state elections of India in 1998. For Uttar Pradesh (■) with 55 015 804 voters and 649 candidates. Goa (◆) with 532 766 voters and 12 candidates. Andhra Pradesh (▲) with 31 829 338 voters and 301 candidates. Haryana (▼) with 7 516 884 voters and 84 candidates and Kerala (*) with 13 036 581 voters and 108 candidates. We obtain results qualitatively comparable to this kind of processes with a pseudo-fractal with 12 generations and 80 candidates, after few iterations (5). The solid lines are squares fits to the data, in the intermediate regions. The slopes are: -1.32 (Uttar Pradesh), -0.97 (Goa), -1.51 (Andhra Pradesh), -2.06 (Haryana), -1.26 (Kerala) and -1.32 (pseudo-fractal).

The differences in the Sznajd simulations on the pseudo-fractal network for the India and Brazilian elections are mainly due to the number of candidates considered for each one. In the latter, the number of candidates is almost 0.005% of the lattice nodes, while for the former it is 0.01%. The fixed point (consensus) is reached the faster, the larger the density of candidates. Owing to this, the comparison of the real elections in India with our simulations were taken into account only five iterations of the Sznajd model on the pseudo-fractal network (20 iterations for Brazilian elections).

3. Conclusions

We studied the behavior of a deterministic scale-free network simulating a spreading of opinion model on it. We solved the Sznajd's model on a growing deterministic scale-free network, obtaining always a consensus of opinion after some runs of the model. The final opinion presents a 1st-order transition, that shows strong hysteresis, as a function of the initial concentration of opinions in the network for rules 2 and 3. The results coincide with the results reported for a stochastic scale-free

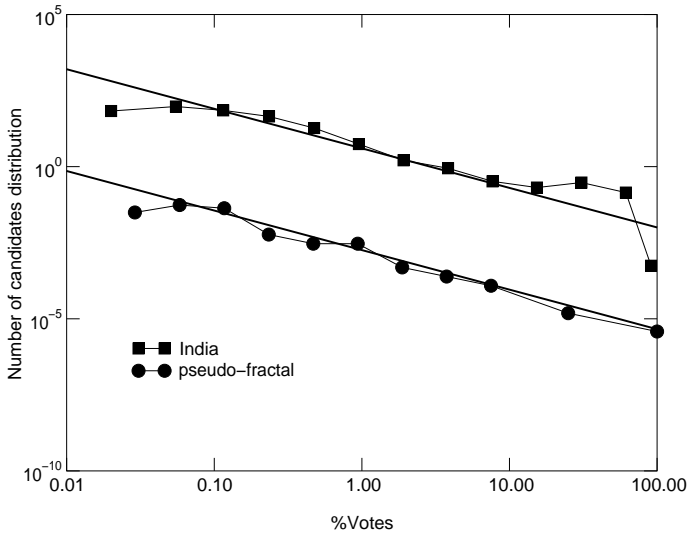


Fig. 10. Voting distribution for India in 1998 (■), compared to the simulation on a pseudo-fractal (●) with 12 generations and 80 candidates, after few iterations (5). The solid lines are guide to the eye with slope -1.3 .

network.¹³ We found that there is no scaling in the finite-size cut-off showing that the system is not critical (Fig. 7). We simulated election processes on the network. The probability of convincing of the model has to be adapted ($\xi = 2$, in Eq. (3)) in order to avoid the differences in the values of the exponents γ of the degree distribution of our deterministic network and the γ exponent of a typical stochastic network (Fig. 4). We obtained the same results, reported for a stochastic scale-free network, but with computation times considerably lower (see Fig. 6). We could use our model to reproduce qualitatively good complex electoral processes, such as the states elections for parliament in Brazil and India.

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